Home Work Problems

1. Since we will use the Gaussian distribution a lot, integrate the following:

$$N = \int_{-\infty}^{\infty} e^{-(x-a)^2} dx$$
$$\langle x \rangle = \frac{1}{N} \int_{-\infty}^{\infty} x\, e^{-(x-a)^2} dx$$
$$\langle x^2 \rangle = \frac{1}{N} \int_{-\infty}^{\infty} x^2\, e^{-(x-a)^2} dx$$
$$\sigma^2 = \left\langle (x - \langle x \rangle)^2 \right\rangle$$

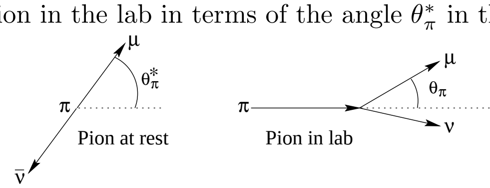
Actually do the integrals, don't just use the cheat sheet.
(Hint: For the first one, try evaluating N^2 and converting to polar coordinates.)

2. Consider a charged pion decaying into a muon plus an antineutrino:

$$\pi^- \to \mu^- + \bar{\nu}_\mu$$

Use $M_{\pi^\pm} = 140\text{MeV/c}^2$, $m_\mu = 106\text{MeV/c}^2$, and $m_{\bar{\nu}} = 0$.

a) In the rest system of the pion, what are the energies and momenta of the muon and antineutrino?
b) For a moving pion with total energy $U_\pi = \gamma M_\pi c^2$ find an expression for the direction, θ_π of the muon relative to the pion in the lab in terms of the angle θ_π^* in the in the pion's rest system.



3. The Tevatron collides protons ($m_p = 0.938$ GeV) at 1 TeV per beam. What is the equivalent proton beam energy required to produce the same center-of-mass energy with a stationary hydrogen target? How fast would you have to drive your new 1.3 ton VW Beetle to have the same kinetic energy as a bunch of 10^{13} protons with this energy? (The speed of sound in air is 330 m/s.)

4. HERA collides 920 GeV protons with 27.5 GeV electrons with zero crossing angle.

a) What is the center-of-mass energy?
b) What is the velocity of the center of mass in the lab system?

5. The Stanford Linear Accelerator is 3.05 km long and can accelerate electrons up to 50 GeV.

a) What is the average accelerating gradient of the rf cavities?
b) For bunches of 4×10^{10} electrons per bunch and a duty cycle of 100 Hz, what is the power transferred to the beam?

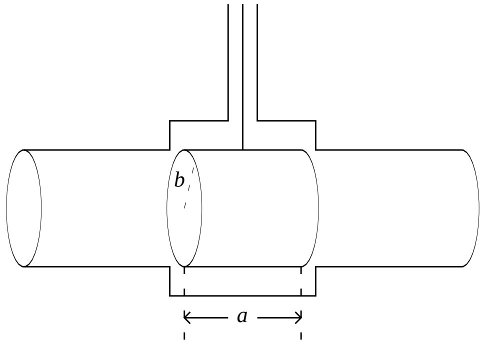
6. An experiment has a 10 cm long liquid hydrogen target with a density of

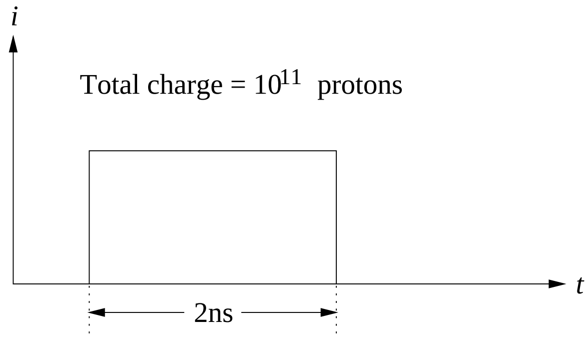
$$\rho = .063 \text{ g/cm}^3.$$

Estimate the interaction rate for p+p collisions for a beam of 10^{13} protons every two minutes. Assume the total cross section is 40 mb. Note: 1 barn = 10^{-24} cm^2 and Avogadro's number is 6×10^{23}.

7. A beam must be bent by an angle $\theta = 1$ mrad by an element of length $\ell = 0.5$ m. Compare the required fields needed for both an electrostatic (parallel plate) and magnetic (dipole magnet) deflectors for proton beams of kinetic energy 10 MeV, 100 MeV, 1 GeV, 10 GeV, and 100 GeV. What voltages would be required if the parallel plates were separated by 5 cm?

8. A relativistic ($\beta \simeq 1$) bunched beam travels down the beam pipe shown below with a ring pickup. The radius of the pipe and pickup is $b = 4$ cm, and the length of the pickup is $w = 5$ cm. Ignore longitudinal fields. Plot the shape of a voltage pulse seen on an oscilloscope with a 50Ω termination. Assume that the coaxial cable is also a 50Ω cable.



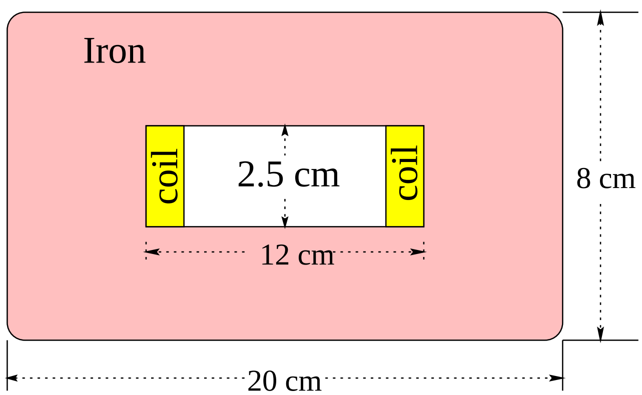


9. The RHIC collider collides fully stripped gold ions ($A = 197$, $Z = 79$) at a total energy of 100 GeV/nucl. per beam. The circumference of each ring is 3834m.
(Assume the mass of a gold ion is 197×0.93113 GeV/c^2.)

a) If the injection energy is 10.5 GeV/nucleon, what is the required swing in revolution frequency during acceleration?

b) If we assume that there are 192 identical dipoles per ring, what is the field at top field? Assume each dipole is 10 m long.

10. Consider a 0.5 m long window-frame dipole with the cross section shown below.



a) Estimate the number of ampere-turns needed to achieve a 0.6 T field in the gap. Assume that the iron is not saturated (i.e., $\mu_r \gtrsim 5000$).

b) Air-cooled copper coils can carry as much as 1.5 A/mm^2; whereas water-cooled copper can carry almost 10 times as much (averaged over conductor and water channels). For the given dimensions of the magnet, would you recommend air-cooled or water-cooled coils. How wide would the gap between the coils be?

c) If the magnet is to be powered by a supply with a maximum current of 1000 A, how many turns should be in the coil.

d) What is the stored energy in the gap.

e) Extra credit: assuming a constant field in the iron estimate the additional energy stored in the iron yoke.

f) Estimate the inductance of the magnet.

11. A lithium lens of length, l, and radius, a, has a current, I, flowing through it with a uniform current density. Consider a beam of antiprotons with momentum, p. What is the focal length of this lens?

12. For general magnetic elements with no xy-coupling and no rf the Transfer matrix for (x, x', z, δ) has the general form:

$$\begin{pmatrix} C & S & 0 & D \\ C' & S' & 0 & D' \\ E & F & 1 & G \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using the symplectic condition $\mathbf{M}^T\mathbf{S}\mathbf{M} = \mathbf{S}$ express E and F in terms of the other elements and show that there is no constraint on the value of G.

13. In Eq. 5.22 verify that

$$e^{\mathbf{J}\mu} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu,$$

where the exponentiation of a square matrix is given by

$$e^{\mathbf{M}} = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{M}^n \quad \text{with} \quad \mathbf{M}^0 = \mathbf{I}.$$

14.

a) Show that the transformation

$$\begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \begin{pmatrix} \beta^{-\frac{1}{2}} & 0 \\ \alpha\beta^{-\frac{1}{2}} & \beta^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} z \\ z' \end{pmatrix},$$

transforms the transfer matrix,

$$\mathbf{M} = e^{\mathbf{J}\mu},$$

into the matrix,

$$\mathbf{N} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}.$$

These new coordinates (ξ, ζ) are sometimes referred to as Floquet or Courant-Snyder coordinates. Note that the ellipse of the Courant-Snyder invariant has been transformed to a circle. Show that the invariant remains unchanged by this transformation.

b) Consider a Gaussian distribution of particles in the new coordinates,

$$f = \frac{N}{2\pi\epsilon} \exp\left(-\frac{\xi^2 + \zeta^2}{2\epsilon}\right).$$

Find the distribution in the old coordinates (z, z'). Evaluate the variances $\sigma_z^2 = \langle (z - \langle z \rangle)^2 \rangle$, and $\sigma_{z'}^2 = \langle (z' - \langle z' \rangle)^2 \rangle$, and the covariance $\sigma_{zz'}^2 = \langle (z - \langle z \rangle)(z' - \langle z' \rangle) \rangle$.

15. Using the invariant

$$\mathcal{W} = \gamma z^2 + 2\alpha z z' + \beta z'^2,$$

show that the Twiss parameters transform from s_1 to s_2 by the matrix transformation

$$\begin{pmatrix} \beta_2 \\ \alpha_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22} + M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \gamma_1 \end{pmatrix}, \quad \text{if}$$

$$\begin{pmatrix} z_2 \\ z_2' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_1' \end{pmatrix}.$$

16. Show that the conversion from rms to 90% and 95% emittances are approximately $\epsilon_{90\%} \simeq 4.605\epsilon_{\text{rms}}$, and $\epsilon_{90\%} \simeq 5.991\epsilon_{\text{rms}}$, for a Gaussian distribution.

17. Verify Eqs. 6.25 and 6.27 of Conte and MacKay. Show that $\alpha_p \sim 1/{Q_{\mathrm{H}}}^2$ if $\mu \lesssim 90°$.

18.

a) What is the maximum possible phase advance in a drift?
b) What is the maximum possible phase advance in a FODO cell?

19. An achromatic bend (the double bend achromat) may be made from two dipoles with a horizontally focusing quadrupole between them. The transfer matrix through the achromat is of the form:

$$\mathbf{M} = \mathbf{B}(\theta)\mathbf{L}(\tfrac{1}{2}\mathbf{Q})(\tfrac{1}{2}\mathbf{Q})\mathbf{L}\mathbf{B}(\theta).$$

a) Use thin the lens approximation for quads and small angle approximation for bends to find the dispersion in the middle of the quad. Write the focal length in terms of the drift length and bend parameters.

$$\text{Hint:} \quad \begin{pmatrix} \eta_c \\ 0 \\ 1 \end{pmatrix} = (\tfrac{1}{2}\mathbf{Q})\mathbf{L}\mathbf{B}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

b) Show that the dispersion is again zero ($\eta = \eta' = 0$) after the bend.

20. Calculate the luminosity in CESR, assuming $\beta_x^* = 1.1$m, $\beta_y^* = 2$cm, $\eta_x^* = 1.1$m, $\sigma_p/p = 6 \times 10^{-4}$, and rms emittances of $\pi\epsilon_x = \pi \times 10^{-7}$m and $\pi\epsilon_y = \pi \times 6 \times 10^{-8}$m. (See Problem 1-1. of Conte & MacKay)

21. Consider a ring made of N identical FODO cells with equally spaced quadrupoles. Assume that the two quadrupoles are both of length l_q, but their strengths may differ. Calculate the natural chromaticities for this machine, and show that for short quadrupoles,

$$\xi_N \simeq -\frac{2\tan\frac{\mu}{2}}{\mu},$$

where μ is the betatron phase advance per cell.

22. For a separated function ring with identical dipole magnets of bending radius, ρ, show that

$$\mathcal{D} = \frac{L\alpha_p}{2\pi\rho},$$

where L is the circumference, and α_p is the momentum compaction.

23. Show that the damping time for the six dimensional phase space volume of a beam is just $\tau_0/8 = U_s/(4U_\gamma)$.

24. According to its design, the Large Hadron Collider (LHC) will be capable of accelerating protons to 7 TeV in each of two rings. The circumference will be 26.7 km, and the arc dipole field at 7 TeV will be 8.33 T.

a) Calculate the critical energy of photons.
b) Calculate the energy loss per turn per proton.
b) Calculate the total power radiated by synchrotron radiation for a beam with an average current of 0.56 A.

25. A light source ring has eight equal double achromat bends (16 dipoles). Each dipole is 2.7 m long, and the circumference is 176 m. The energy of the beam is 2.5 GeV.

a) Calculate the critical energy of photons radiated in the dipoles.
b) Calculate the total energy lost per turn.
c) Calculate the momentum compaction of the ring.
d) Calculate the damping times τ_x, τ_y, and τ_u.

26. Consider a helical dipole magnet twisted through a full 360° in a length λ with approximate transverse field components given by

$$B_x \simeq -B_0 \left\{ \left[1 + \frac{k^2}{8}(3x^2 + y^2)\right] \sin kz - \frac{k^2}{4} xy \cos kz \right\}$$
$$B_y \simeq B_0 \left\{ \left[1 + \frac{k^2}{8}(x^2 + 3y^2)\right] \cos kz - \frac{k^2}{4} xy \sin kz \right\}$$
$$B_z \simeq -B_0 k \left[1 + \frac{k^2}{8}(x^2 + y^2)\right] (x \cos kz + y \sin kz)$$

where the pitch is $k = \frac{2\pi}{\lambda}$.

a) Show that this parameterization satisfies Maxwell's equations to second order in the transverse dimensions.

b) Ignoring the transverse dependence of the field

$$(B_x \simeq -B_0 \sin kz, \quad B_y \simeq B_0 \cos kz, \quad \text{and} \quad B_z \simeq 0)$$

find an approximation for the trajectory of a particle through the magnet in the paraxial approximation.

(This type of magnet can be used in helical undulators for FEL's and in Siberian snakes and rotators for spin rotation of polarized beams.)

27. A gold ($^{197}\mathrm{Au}^{+79}$) beam passes through a 1 mm thick $\mathrm{Al_2O_3}$ flag, at a location in the beam line where the Twiss parameters are $\beta_\mathrm{H} = \beta_\mathrm{V} = 6$ m and $\alpha_\mathrm{H} = \alpha_\mathrm{V} = 0$. Multiple Coulomb scattering takes place in the flag adding to the angular divergence of the beam. A good approximation of the deflection is a Gaussian distribution with an rms angle given by

$$\bar{\theta} = \sqrt{\langle \theta^2 \rangle} \simeq z \frac{20 \text{ MeV/c}}{p\beta} \sqrt{\frac{x}{L_\mathrm{rad}}} \left(1 + \frac{1}{9} \log_{10} \frac{x}{L_\mathrm{rad}}\right),$$

where z is the beam particle's charge, x is the thickness of the flag expressed in $[\mathrm{g} \cdot \mathrm{cm}^{-2}]$. Assume $L_\mathrm{rad} = 24\ \mathrm{g} \cdot \mathrm{cm}^{-2}$ and $\rho_{\mathrm{Al_2O_3}} = 3.7\ \mathrm{g} \cdot \mathrm{cm}^{-3}$. The gold beam has a total energy of 10 GeV/nucleon with $M_\mathrm{Au}/A = 0.93113$ GeV/nucleon.

a) Evaluate $\bar{\theta}$.

b) The beam has a normalized emittance $\epsilon^N_{95\%} = 10\ \mu$m just before the flag. Estimate the blowup in emittance from the flag.

28. Consider a ring with a thin rf cavity whose linear transfer matrix just after the cavity is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & Q & 1 \end{pmatrix} \begin{pmatrix} C & S & 0 & D \\ C' & S' & 0 & D' \\ E & F & 1 & G \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

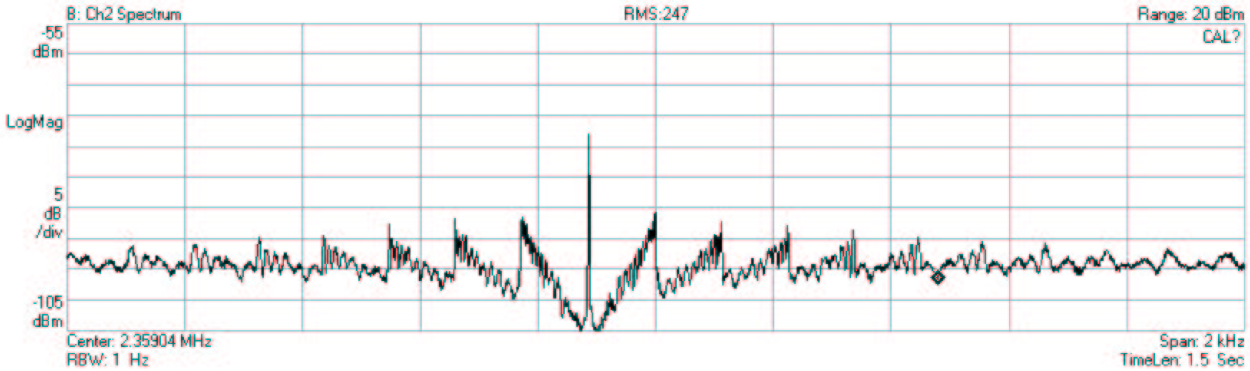
Show that the dispersion functions are still given by

$$\eta = \frac{(1 - S')D + SD'}{2(1 - \cos\mu)}, \qquad (5.87[\mathrm{CM}]))$$
$$\eta' = \frac{(1 - C)D' + C'D}{2(1 - \cos\mu)}. \qquad (5.88[(\mathrm{CM}])$$

Hint: The eigenvector equation of Eq. 5.86[CM] must be modified to allow for momentum compaction:

$$\mathbf{M} \begin{pmatrix} \eta \\ \eta' \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \eta \\ \eta' \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Delta L \\ \Delta\delta \end{pmatrix}.$$

29. Given the following Schottky spectrum for RHIC with $^{197}Au^{79+}$ ions (fully stripped) where only a single storage cavity was powered, calculate the rf voltage in the cavity. Assume the following parameters for RHIC: $\gamma_{\rm tr} = 22.8$, circumference of $L = 3833.845$ m, harmonic number $h = 7 \times 360 = 2520$, and $U_s = 100$ GeV/nucleon at fixed energy.



30. Show that the rf power loss in the conducting walls of a cavity is given by Eq. (9.16) of Conte and MacKay.

31. Consider a box shaped resonant cavity with a square cross section of width, w, in the transverse directions and length, l, in the longitudinal dimension. Calculate the resonant frequency and Q of the lowest order TM mode.

32.

a) Calculate the synchrotron tune for RHIC for fully stripped gold ions ($^{197}Au^{79+}$)

$$\begin{aligned}
\gamma_{\rm inj} &= 10.4 \\
\gamma{\rm tr} &= 22.8 \\
L &= 3834{\rm m} \\
h &= 360 \\
\phi_s &= 0^\circ \\
mc^2 &= 197 \times 0.93113 \text{ GeV} \\
Z &= 79 \quad \text{(protons)} \\
A &= 179 \quad \text{(neutrons + protons)} \\
V_{\rm rf} &= 300 \text{ kV}
\end{aligned}$$

b) What is the syncrhrotron frequency?

c) For a synchronous phase of $\phi_s = 5.5^\circ$, how much energy does the synchronous particle gain per turn?

d) How long would it take to accelerate to $\gamma = 107.4$ (100 GeV/nucleon)? Assume that the phase jump at transition has been performed correctly (i.e., ignore it).

e) Plot the synchrotron frequency as a function of energy.

33.

a) Using Eqs. (4.6 & 4.7 of Conte and MacKay), show that the field in the polar coordinate representation is

$$B_r = B_0 \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n \left[a_n \cos((n+1)\theta) + b_n \sin((n+1)\theta)\right]$$

$$B_\theta = B_0 \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n \left[b_n \cos((n+1)\theta) - a_n \sin((n+1)\theta)\right].$$

b) Show that the vector potential

$$
\begin{aligned}
A_r &= 0 \\
A_\theta &= 0 \\
A_z &= B_0 \sum_{n=0}^{\infty} \frac{a}{n+1} \left(\frac{r}{a}\right)^{n+1} \left[a_n \sin((n+1)\theta) - b_n \sin((n+1)\theta)\right].
\end{aligned}
$$

leads to the components given in part (a).